Payoff and Revenue Inequivalence in Repeated Auction with Time-Varying Number of Bidders

Yuma Fujimoto
CyberAgent
Tokyo, Japan
fujimoto.yuma1991@gmail.com

Kaito Ariu
CyberAgent
Tokyo, Japan
kaito_ariu@cyberagent.co.jp

Kenshi Abe
CyberAgent
Tokyo, Japan
abe_kenshi@cyberagent.co.jp

ABSTRACT

Ad auctions employ several mechanisms, primarily first- and second-price auctions. One primary question is which mechanism is preferred from the perspective of bidders' payoffs and the seller's revenue. An answer is celebrated revenue equivalence, where both the payoff and revenue are equal in equilibrium between different mechanisms. In reality, however, this equilibrium does not always hold, partly because auction environments, such as the number of bidders, vary over time. This study questions whether and how payoff/revenue equivalence breaks down as the number of bidders varies over time. Interestingly, any time-varying patterns lead to payoff losses for bidders. On the other hand, they result in revenue gains or losses for the seller, depending on the case. Our theorems evaluate the payoff/revenue gaps across a broad range of value distributions, including representative examples such as power, exponential, and Pareto distributions. Our experiments further visualize and strengthen our theorems. This paper discovered and provided theoretical insight into complex phenomena that could occur in reality.

KEYWORDS

Auction, Revenue Equivalence, Learning in Games, Time-Varying Game

1 INTRODUCTION

Auction is one of the most representative buying and selling systems [30]. This system is applied to ad auctions, where a seller (i.e., publisher) provides an item (i.e., ad space), while multiple buyers (i.e., advertisers) compete for the space to post their ads. Each buyer observes the value of an item and decides the bidding price as their strategy. A bidder who bids the highest price wins the item. The winner's payment for the item depends on the details of the auction mechanism. In the application of advertisement, first-price and second-price auctions are normally adopted: the former requires the winner to pay the highest price (i.e., the price the winner bids), whereas the latter the second-highest price (i.e., the highest price the others bid).

One primary motivation to study auction theory is to determine which auction mechanism (e.g., first-price and second-price) maximizes the payoff of bidders and the revenue of sellers. A celebrated auction theory has proven revenue equivalence, where revenue is equal in the equilibrium bidding between different auction mechanisms. This revenue equivalence immediately leads to payoff equivalence. This payoff/revenue equivalence holds when the value of an item is symmetric, private, and independent among bidders. Empirical ad auction data imply that a phenomenon like revenue equivalence is observed; the price levels converge between first-price and second-price auctions [13]. However, ad auction platforms have shifted their mechanism from second-price to first-price [1, 6, 27]. Payoff/Revenue equivalence is a significant phenomenon related to the real-world economy.

1.1 Time-Varying Environment

Although payoff/revenue equivalence requires equilibrium in bidding behavior, such an equilibrium is not always achieved in reality. One reason is that some environmental parameters of auctions, such as the number of bidders, vary over time (see Fig. 1-A). For example, the demand for advertising space in hotels would differ between weekdays and holidays, and the number of bidders is expected to change following a weekly cycle. Furthermore, the number of bidders fluctuates due to randomness. When the number of bidders is larger, the competition among them is more intense, leading to a higher payment. On the other hand, the smaller number of bidders leads to a lower payment. Such periodicity and randomness are observed in empirical auction data [13, 31, 32].

Non-equilibrium bidding caused by such a time-varying number of bidders can lead to the breaking of payoff/revenue equivalence, even if value distribution is always symmetric, independent, and private among bidders. In first-price auctions, bidders aim to bid the equilibrium (optimal) payment itself, which varies depending on the number of auction participants. If the number of auction participants also varies, they cannot keep the equilibrium and can only track the equilibrium by learning (see the orange line in Fig. 1-B). On the other hand, in the second-price auctions, bidders only have to take truthful bidding, independent of the number of auction participants. Even if the number of auction participants varies, the payment automatically adjusts to the optimal one (see the gray line in Fig. 1-B). Therefore, one significant issue is whether and how the difference between non-equilibrium in first-price auctions and equilibrium in second-price auctions breaks the payoff/revenue equivalence down.

1.2 Dynamical Systems Approach

This study focuses on the dynamical systems analysis to capture such non-equilibrium bidding in first-price auctions. Indeed, this dynamical systems analysis has succeeded in understanding non-equilibrium behaviors, such as cycling around the equilibrium, in the context of learning in games [2, 3, 10, 22, 23, 25, 29], which is deeply related to repeated auctions. In time-varying games, where the rule of the game varies similarly to this study, understanding the dynamical systems becomes even more important [8, 9, 11].

Proc.@ of the Adaptive and Learning Agents Workshop (ALA 2026), Aydeniz, Delgrange, Mohammedalamen, Yang (eds.), May 25 – 26, 2026, Paphos, Cyprus, https://alaworkshop2026.github.io/. 2026.

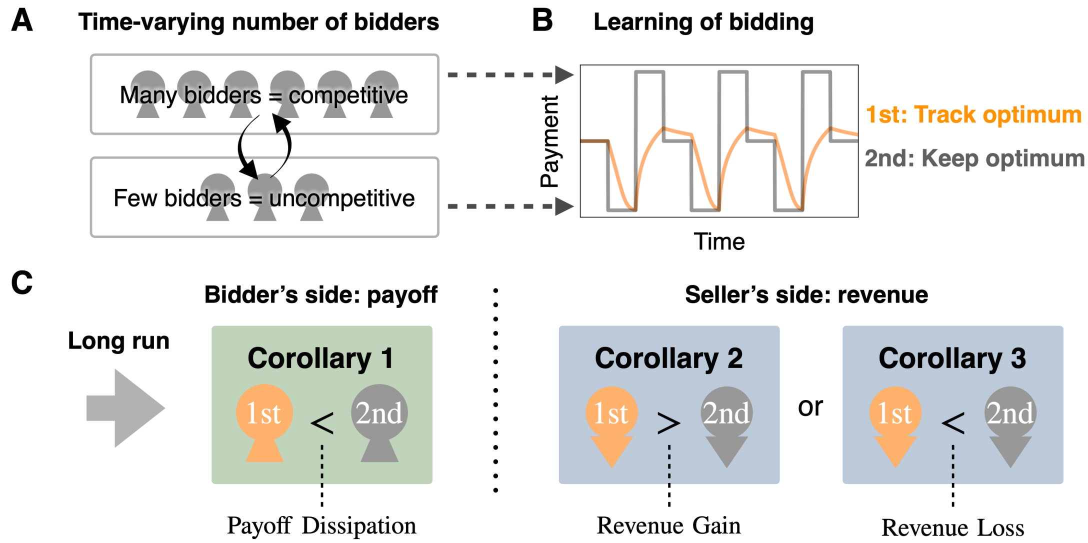

Figure 1: Overview of this study. (A). This study considers an auction where the number of bidders varies over time. When more (fewer) bidders participate, the auction is more (less) competitive. (B). Learning of bidding is discussed for both the first-price (1st: orange line) and second-price (2nd: gray) auctions. Here, note that in second-price auctions, bidders keep optimum payment despite the time-varying number of bidders. In first-price auctions, however, they have to track the optimal payment (bidding) by learning and thus cannot reach the optimum under the time-varying number of bidders. (C). We analyze the long run of learning. For bidders, the long-run payoff is compared between first-price and second-price auctions. In Cor. 4.2, we show that the first-price auctions always generate a dissipation in the payoff (green panel) compared to second-price auctions. On the other hand, for the seller, the long-run revenue is discussed. In Cors. 4.4 and 4.5, we show that first-price auctions generate a gain (left blue panel) or loss (right) in the revenue.

In the context of repeated auction, a dynamical systems approach has been taken to analyze non-convergence behavior [5, 19, 26]. The most relevant study [12] considers the dynamical systems in time-varying auctions, where the value distribution varies, not the number of bidders. Although the previous studies [24, 28, 30] have discussed payoff/revenue equivalence at the moment when the equilibrium is achieved, this study discusses it based on the long-run behavior of such non-equilibrium dynamics.

1.3 Our Contribution

To the problems about the time-varying number of bidders, our contributions are summarized as follows.

- **We propose and analyze learning dynamics in time-varying auctions.** We extend the classical equilibrium analysis to learning dynamics by gradient ascent, which is guaranteed to converge to the time-invariant equilibrium.
- **We prove whether and how payoff/revenue equivalence is broken by the time-varying number of bidders.** Surprisingly, we find that bidders suffer the loss of their payoffs in first-price auctions independent of how the number varies over time (see the green panel in Fig. 1-C). For the seller, however, the time-varyingness leads to both the loss and gain of the revenue depending on the case (see the blue panels in Fig. 1-C). These findings are proven for a wide range of value distributions, including power, exponential, and Pareto distributions, all of which are majorly analyzed in auction theory.
- **We numerically demonstrate payoff/revenue inequivalence.** The simulations for the power distributions visualize the results of our theory. Moreover, those for the beta distributions support that the results are applicable even beyond the assumptions made for the sake of analysis.

2 PRELIMINARY: STATIC ENVIRONMENT

First, we review classical auction theory, which supposes a static environment.

2.1 Setting

We consider $n \in \mathbb{N}$ bidders. They are offered an item, which value is $x_i \in \mathbb{R}$ for bidder $i \in \{1, \cdots, n\}$. We consider that each bidder bids $b_i \in \mathbb{R}$ for the item. In any type of auction, if bidder i bids the highest price, i.e., $b_i = \max_j b_j$, the bidder receives the item in exchange for some payment. Here, this payment changes depending on the type of auction. Representatively, the first-price auction requires $b^{1\text{st}} = \max_j b_j$, while the second-price auction does $b^{2\text{nd}} = \max_{j \neq i} b_j$. Thus, the payoff of each bidder i is expressed as

$$u_i^{1\text{st}}(b_i|v_i) = (v_i - b^{1\text{st}})\mathbb{I}[b_i = \max_j b_j], \tag{1}$$

$$u_i^{2\text{nd}}(b_i|v_i) = (v_i - b^{2\text{nd}})\mathbb{I}[b_i = \max_j b_j], \tag{2}$$

for the cases of first-price and second-price auctions, respectively.

We further consider that each bidder's value independently follows the same probability distribution, i.e., $x_i \sim f$, with its cumulative distribution, denoted as F, continuous. This is called "symmetric bidders" with "independent values". We also assume that each bidder can observe only its value x_i, called "private values". Thus, the strategy of each bidder is to determine its bidding to the observed value, i.e., $b_i(x_i)$.

2.2 Symmetric Bayesian Nash equilibrium

Following the methods in the celebrated studies [24, 28], we can obtain the symmetric Bayesian Nash equilibrium [14, 15] as $b_i = b^*$ for all i such that

$$b^*(x) = x - \frac{1}{F(x)^{n-1}} \int_0^x F(z)^{n-1} \mathrm{d}z. \tag{3}$$

Since a winner pays its bidding, the payment is also $b^*(x)$ in the equilibrium. Here, note that the equilibrium bidding (payment) depends on environmental parameters like population size n and value distribution function F.

In the second-price auctions, the Bayesian Nash equilibrium is "truthful bidding", i.e., $b_i = b^{\mathrm{TB}}$ for all i such that $b^{\mathrm{TB}}(x) = x$. Note that this truthful bidding does not depend on environmental parameters such as n and F. However, the expected payment by the truthful bidding depends on the environment and corresponds to that in the first-price auction, i.e., $b^*(x)$.

3 TIME-VARYING NUMBER OF BIDDERS

This study considers a dynamic environment where the number of bidders varies over time, i.e., $n(t) \in \mathbb{N}$ for $t \in [0, T]$. In second-price auctions, truthful bidding is resistant to such a dynamic environment and keeps the equilibrium payment

$$b(x; m^*(t)) := x - \frac{1}{F(x)^{m^*(t)}} \int_0^x F(z)^{m^*(t)} \mathrm{d}z. \tag{4}$$

On the other hand, in first-price auctions, bidders would not achieve their optimal bids (payments) because they cannot observe the dynamic number of bidders. In general, we assume that a bidder estimates the true number of others $m^*(t) = n(t) - 1$ as a real number $m(t) \in \mathbb{R}$. The optimal bidding for this bidder is given by replacing the true value $m^*(t)$ with their estimation $m(t)$ in Eq. (4) as

$$b(x; m(t)) := x - \frac{1}{F(x)^{m(t)}} \int_0^x F(z)^{m(t)} \mathrm{d}z. \tag{5}$$

3.1 Learning of Equilibrium Bidding

First, let $w_{m^*}(m', m)$ be the expected payoff of the focal bidder who uses $b(x; m')$ when all the others use $b(x; m)$ under the true number m^*, which is described as

$$w_{m^*}(m', m) := \int_0^{x_{\mathrm{M}}} (x - b(x; m')) f(x) F(x')^{m^*} \mathrm{d}x. \tag{6}$$

Here, we defined $x' = x'(x, m, m')$ such that $b(x'; m) = b(x; m')$, meaning that when the focal bidder whose estimation is m' observes the item value x, it bids as if it observes x' for the others whose estimations are m.

Next, let us formulate how each bidder learn their estimation $m(t)$. Extending the symmetric Bayesian Nash equilibrium to a non-equilibrium case, suppose that all the bidders symmetrically estimate the number of others as $m(t)$. Suppose that each bidder observes the local gradient of their own expected payoff and incrementally updates their estimation to receive more payoff. This is formulated as gradient descent-ascent, whose continuum limit is

$$\dot{m}(t) = \left. \frac{\partial w_{m^*(t)}(m', m(t))}{\partial m'} \right|_{m'=m(t)}. \tag{7}$$

Such a continuum limit is known to well-approximate behavior with a small learning rate. The learning dynamics of Eq. (7) are calculated as

$$\dot{m}(t) = -\frac{m(t) - m^*(t)}{m(t)(n(t) - m(t))} (T_{m(t)} - S_{m(t),n(t)}), \tag{8}$$

(see Appendix B.1 for the detailed calculation). Here, we used the notations of

$$T_{m(t)} := -\int_0^{x_{\mathrm{M}}} F(x)^{m(t)} \log F(x) \mathrm{d}x, \tag{9}$$

$$S_{m(t),n(t)} := \frac{1}{n(t) - m(t)} \int_0^{x_{\mathrm{M}}} F(x)^{m(t)} - F(x)^{n(t)} \mathrm{d}x. \tag{10}$$

Remark on gradient dynamics: Modeling each bidder as updating a single variable via gradient ascent is an effective approach for extracting simple insights from complex auction phenomena. Indeed, recent work similarly employs single-variable gradient dynamics to characterize non-convergent behaviors in autobidding systems [26]. Furthermore, our gradient-based update rule serves as the continuous-time limit of mean-based learning algorithms, discussed in prior literature [5]. While the gradient dynamics require each bidder to observe the local gradient of their payoff, this is called "full feedback" setting and remains a standard assumption in the literature on learning in games including auctions. Although alternative approaches might suppose that each bidder maintains a belief distribution over their number [16], our dynamics offer the robust property of guaranteed convergence to the symmetric Nash equilibrium in static environments (see Appendix B.2).

Remark on symmetric bidders: Our assumption that all bidders maintain the same estimation variable $m(t)$ (i.e., symmetric bidders) is justified from several perspectives. First, this symmetry is a standard setting in classical auction theory, and our framework serves as a natural extension of the symmetric Bayesian Nash equilibrium into the non-equilibrium regime. Second, since all bidders share the common environmental feedback (the true number $n(t)$), their estimates are expected to synchronize over time, even if they originate from heterogeneous estimates. Finally, the assumption is theoretically grounded in the framework of "adaptive dynamics" from evolutionary game theory [7, 17]. This framework establishes that, under small update steps, the evolution of a heterogeneous population can be effectively approximated by a representative agent. In the context of repeated auctions, the framework offers a significant analytical advantage: it approximates complex behavior of multi-agent learning well by effectively filtering out heterogeneity, for example, arising from learning noise.

3.2 Calculation of Payoff

Furthermore, let us analyze the expected payoff of the bidders in first-price and second-price auctions. Because all the bidders use $b(x; m(t))$ in the first-price auctions, their expected payoff is calculated as

$$w^{\text{1st}}_{m^*(t)}(m(t)) := w_{m^*(t)}(m(t), m(t)) = S_{m(t),n(t)}, \tag{11}$$

(see Appendix B.3 for the detailed calculation of the second equality). In second-price auctions, all the bidders obtain the Nash equilibrium payoff, which is given by the case of $m(t) = m^*(t)$. Thus, the equilibrium payoff is calculated as

$$w^{\text{2nd}}_{m^*(t)} := w^{\text{1st}}_{m^*(t)}(m^*(t)) = S_{m^*(t),n(t)}. \tag{12}$$

By the property of $S_{m(t),n(t)} > 0$, it trivially holds that the bidders always enjoy positive payoffs.

3.3 Calculation of Revenue

Last, we discuss the seller's revenue. The social welfare, i.e., the expected maximum value that one item generates for $n(t)$ bidders, is described as

$$\text{SW}_{m^*(t)} := x_{\text{M}} - \int_0^{x_{\text{M}}} F(x)^{n(t)} \text{d}x. \tag{13}$$

In first-price auctions, where all the bidders use $b(x; m(t))$, the expected revenue is defined as

$$R^{\text{1st}}_{m^*(t)}(m(t)) := \text{SW}_{m^*(t)} - n(t) w^{\text{1st}}_{m^*(t)}(m(t)). \tag{14}$$

This means that the seller's revenue $R^{\text{1st}}_{m^*(t)}(m(t))$ conflicts with the bidders' total payoff $n(t) w^{\text{1st}}_{m^*(t)}(m(t))$. In second-price auctions, where all the bidders use $b(x; m^*(t))$, the expected revenue is calculated as

$$R^{\text{2nd}}_{m^*(t)} := R^{\text{1st}}_{m^*(t)}(m^*(t)) = \text{SW}_{m^*(t)} - n(t) w^{\text{2nd}}_{m^*(t)}. \tag{15}$$

4 ANALYSIS OF LONG-RUN BEHAVIOR

For convenience in comparing the expected payoff and revenue between first-price and second-price auctions, we define

$$\Delta w_{m^*(t)}(m(t)) := w^{\text{1st}}_{m^*(t)}(m(t)) - w^{\text{2nd}}_{m^*(t)}, \tag{16}$$

$$\Delta R_{m^*(t)}(m(t)) := R^{\text{1st}}_{m^*(t)}(m(t)) - R^{\text{2nd}}_{m^*(t)} \tag{17}$$

$$= -n(t) \Delta w_{m^*(t)}(m(t)). \tag{18}$$

Here, Δ denotes the difference between first-price and second-price auctions.

This study considers the time series of the number of bidders $n(t)$ in the whole time $t \in [0, T]$. Following the time series of $n(t)$, the time series of m is generated by Eq. (7). We evaluate the long-run behavior of $\Delta w_{m^*(t)}(m(t))$ and $\Delta R_{m^*(t)}(m(t))$ by defining

$$\Delta \bar{w}(T) := \frac{1}{T} \int_0^T \Delta w_{m^*(t)}(m(t)) \text{d}t, \tag{19}$$

$$\Delta \bar{R}(T) := \frac{1}{T} \int_0^T \Delta R_{m^*(t)}(m(t)) \text{d}t. \tag{20}$$

Here, "bar" means taking the time average. In the same way, we also define $\bar{w}^{\text{1st}}(T)$, $\bar{w}^{\text{2nd}}(T)$, $\bar{R}^{\text{1st}}(T)$, and $\bar{R}^{\text{2nd}}(T)$.

Meaning of $\Delta\bar{w}(\infty)$ and $\Delta\bar{R}(\infty)$: The rest of this paper focuses on evaluating whether $\Delta\bar{w}(\infty)$ and $\Delta\bar{R}(\infty)$ are positive or negative, respectively. This $\Delta\bar{w}(\infty)$ is a key indicator in determining which of the first-price and second-price auctions is preferable for the bidders, while $\Delta\bar{R}(\infty)$ is for the seller.

- $\Delta\bar{w}(\infty) = 0$ ($\Delta\bar{R}(\infty) = 0$): Long-run payoff (revenue) equivalence holds. Each buyer's payoff (The seller's revenue) is equal between first-price and second-price auctions in the time average.
- $\Delta\bar{w}(\infty) > 0$ ($\Delta\bar{R}(\infty) > 0$): Long-run payoff (revenue) equivalence is broken. Time-varying number of bidders generates a gain in the payoff of each buyer (the revenue of the seller).
- $\Delta\bar{w}(\infty) < 0$ ($\Delta\bar{R}(\infty) < 0$): Long-run payoff (revenue) equivalence is broken. Time-varying number of bidders generates a loss in the payoff of each buyer (the revenue of the seller).

Independence between $\Delta\bar{w}(\infty)$ and $\Delta\bar{R}(\infty)$: $\Delta R_{m^*(t)}(m(t)) = -n(t)\Delta w_{m^*(t)}(m(t))$ obviously means that at the moment t, the revenue $\Delta R_{m^*(t)}(m(t))$ conflicts with the payoff $\Delta w_{m^*(t)}(m(t))$. However, the long-run revenue $\Delta\bar{R}(\infty)$ does not always conflict with the long-run payoff $\Delta\bar{w}(\infty)$. This is because $\Delta\bar{R}(\infty)$ is not proportional to $\Delta\bar{w}(\infty)$. Thus, we need to evaluate both $\Delta\bar{w}(\infty)$ and $\Delta\bar{R}(\infty)$ separately.

4.1 Assumptions

Before discussing time-average revenue equivalence, we provide several assumptions. First, Asm. 1 considers that bidders do not reach equilibrium in almost all times $t \in [0, T]$. This assumption is necessary to remove the cases where the number of bidders does not change at all.

Assumption 1 (Non-equilibrium almost everywhere). *It does not hold that $m(t) = m^*(t)$ almost everywhere in $t \in [0, T]$.*

Interpretation: We now interpret Asm. 1. $m(t) = m^*(t)$ means that the bidders can accurately predict the number of others. Thus, learning has reached equilibrium at the moment t. This assumption implies that learning does not reach equilibrium, i.e., $\dot{m}(t) \neq 0$, at almost all times. When an auction environment continues to change over time, learning is considered to rarely reach the equilibrium. In conclusion, the assumption is reasonable in reality.

We also consider value distributions satisfying Defs. 1 and 2. The former is necessary to evaluate long-run payoff and revenue in our theorems, while the latter is to obtain our corollaries. Analysis in auction theory often assumes the structure of distributions, such as monotonic hazard rate or regularity [4, 18, 21]. Our class balances mathematical tractability with wide applicability. Later, we prove that it includs many types of distributions, i.e., Exms. 1-3, that are frequently analyzed in the context of auction theory [20].

Definition 1 (Inverse polynomial distribution). *F is an inverse polynomial distribution if and only if the inverse function of F, i.e., $F^{-1} =: P$ can be described as $P(F) = \sum_{k \in K} a_k F^k$ for some countable set K where $k > 0$ for all $k \in K$.*

Definition 2 (Positive coefficients). *An inverse polynomial distribution F is defined to have only positive coefficients if and only if $a_k > 0$ for all $k \in K$.*

Example 1 (Power distribution). *The power distributions, $F(x) = x^{\alpha}$ over $x \in [0,1]$ where $\alpha > 0$, are inverse polynomial with positive coefficients.*

Proof. Assume $F(x) = x^{\alpha}$ over $x \in [0,1]$ where $\alpha > 0$, and then we obtain $P(F) = F^{1/\alpha}$. Thus, it is sufficient to set $K = \{1/\alpha\}$ and $a_{1/\alpha} = 1$. □

Example 2 (Exponential distribution). *The exponential distributions, $F(x) = 1 - \exp(-\lambda x)$ over $x \in [0,\infty)$ where $\lambda > 0$, are inverse polynomial with positive coefficients.*

Proof. Assume $F(x) = 1 - \exp(-\lambda x)$ over $x \in [0,\infty)$ where $\lambda > 0$, and then we obtain

$$P(F) = -\frac{1}{\lambda}\log(1-F) = -\frac{1}{\lambda}\sum_{k=1}^{\infty} -\frac{1}{k}F^k. \tag{21}$$

Thus, it is sufficient to set $K = \mathbb{N}$ and $a_k = \frac{1}{\lambda k}$ for all $k \in K$. □

Example 3 (Pareto distribution). *The Pareto distribution, $F(x) = 1 - (1+x)^{-2}$ over $x \in [0,\infty)$, is inverse polynomial with positive coefficients.*

Proof. Assume $F(x) = 1 - (1+x)^{-2}$ over $x \in [0,\infty)$, and then we obtain

$$P(F) = (1-F)^{-\frac{1}{2}} - 1. \tag{22}$$

The derivative of k-th order is calculated as

$$P^{(k)}(F) = (1-F)^{-k-\frac{1}{2}}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} \quad \Rightarrow P^{(k)}(0) > 0, \tag{23}$$

where we use the gamma function Γ. By using the property of $a_k = P^{(k)}(0)/\Gamma(k+1)$, it is sufficient to set $K = \mathbb{N}$ and $a_k = \Gamma(k+\frac{1}{2})/(\Gamma(\frac{1}{2})\Gamma(k+1))$. □

4.2 Long-Run Payoff Inequivalence

Let us discuss long-run payoff equivalence. Using the expression of inverse polynomial functions, Thm. 4.1 evaluates $\Delta\bar{w}(\infty)$ by the term whose sign is independent of t. Furthermore, when the inverse polynomial function has only positive coefficients, this theorem derives Cor. 4.2, determining the sign of $\Delta\bar{w}(\infty)$.

The following theorem holds for the difference in time-average payoffs between first-price and second-price auctions (see Appendix C.1 for the full proof).

Theorem 4.1 (Time-average payoff). *For an inverse polynomial function $F(x)$, the difference in time-average payoffs between first-price and second-price auctions is evaluated as*

$$\Delta\bar{w}(\infty) = -\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{Dissipation}(t)\mathrm{d}t, \tag{24}$$

$$\mathrm{Dissipation}(t) = \sum_{k\in K}\frac{a_k k(m(t)-m^*(t))^2}{(m(t)+k)^2(m^*(t)+k)(n(t)+k)}. \tag{25}$$

Proof Sketch. First, notice that if $f(x)$ exists within the integral, $T_{m(t)}$ and $S_{m(t),n(t)}$ are computable for any $F(x)$. If $F(x)$ is inverse polynomial, $\sum_{k\in K} a_k F(x)^k = x \Leftrightarrow f(x)\sum_{k\in K} k a_k F(x)^{k-1} = 1$ holds. By inserting $f(x)\sum_{k\in K} k a_k F(x)^{k-1} = 1$, we can compute the integrals of $T_{m(t)}$ and $S_{m(t),n(t)}$ and obtain

$$\Delta w_{m^*(t)}(m(t)) = m(t)\dot{m}(t) - \mathrm{Dissipation}(t). \tag{26}$$

Here, the first term in the RHS (i.e., $m(t)\dot{m}(t)$) is negligible in its time average by the boundedness of $m(t)$. Thus, we have proved the theorem. □

Corollary 4.2 (Payoff dissipation). *If an inverse polynomial function $F(x)$ has only positive coefficients, $\Delta\bar{w}(\infty) < 0$ holds under Asm. 1.*

Proof. If $F(x)$ has only positive coefficients, $a_k > 0$ holds for all k, and $\mathrm{Dissipation}(t)$ in Eq. (24) is always positive. We have proved $\Delta\bar{w}(\infty) < 0$. □

Remark on the corollary: The inequality $\Delta\bar{w}(\infty) < 0$ is surprising. This inequality holds for arbitrary time series of $n(t) = m^*(t) + 1$, which exist infinitely. Furthermore, $m(t)$ is determined by the time series of $n(t)$. Thereby, both the cases of $m(t) - m^*(t) > 0$ and < 0 appear in complex patterns in the time series of $m(t) - m^*(t)$. Since $\Delta w_{m^*(t)}(m(t)) > 0$ (resp. < 0) holds for $m(t) - m^*(t) < 0$ (resp. > 0), it is difficult to directly judge whether $\Delta\bar{w}(\infty)$ is positive or negative. In the proof, we discover that $\Delta w_{m^*(t)}(m(t))$ is divided into the bounded term (i.e., $m(t)\dot{m}(t)$) and the squared term of $m(t) - m^*(t)$ (i.e., $\mathrm{Dissipation}(t)$). This leads to the inequality independent of the detailed time series of $n(t)$.

Non-equilibrium leads to dissipation: We also discuss the meaning of Thm. 4.2 deeply. In $\Delta w_{m^*(t)}(m(t))$, the dissipation term means that learning in first-price auctions generates a payoff loss. This dissipation term is proportional to $(m(t) - m^*(t))^2$. Here, recall that $m(t) = m^*(t) \Leftrightarrow m(t) - m^*(t) = 0$ holds in the equilibrium of the learning dynamics. In light of this equilibrium, $|m(t) - m^*(t)|$ indicates the degree of non-equilibrium in learning. Thus, the non-equilibrium results in the dissipation.

4.3 Long-Run Revenue Inequivalence

Let us also discuss long-run revenue equivalence. Thm. 4.3 evaluates $\Delta\bar{R}$ by the term whose sign is independent of t. Furthermore, Cors. 4.4 and 4.5 show that the sign of $\Delta\bar{R}$ is determined depending on k in the expression of inverse polynomial functions.

The following theorem holds for the difference in time-average revenues between first-price and second-price auctions (see Appendix C.2 for the full proof).

Theorem 4.3 (Time-average revenue). *For an inverse polynomial function $F(x)$, the difference in time-average revenues between first-price and second-price auctions is evaluated as*

$$\Delta\bar{R}(\infty) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{RevenueGap}(t)\mathrm{d}t, \tag{27}$$

$$\mathrm{RevenueGap}(t) = \sum_{k\in K}\frac{-a_k k(k-1)(m(t)-m^*(t))^2}{(m(t)+k)^2(m^*(t)+k)(n(t)+k)}. \tag{28}$$

Proof sketch. To obtain independency of the time series of $n(t)$, we focus on generating the square of $m(t) - m^*(t)$. Now,

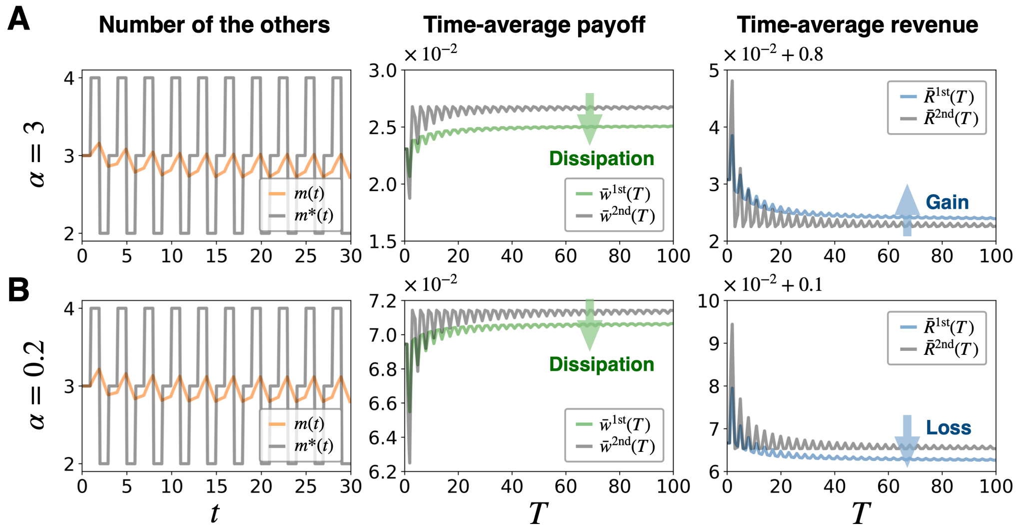

Figure 2: Simulations for the power distribution (Exm. 1). We plot the time series of the number of the other bidders (the left panels), time-average payoff (center), and time-average revenue (right). In all the panels, the colored (resp. gray) lines indicate the time series for first-price (resp. second-price) auctions. For the numerical calculation, we used the Runge-Kutta fourth-order method with the step size of $1/10$, and learning is accelerated 200 times. When we calculate the integrals of $S_{m(t),n(t)}$ and $T_{m(t)}$, we approximate $F(x)$ by a step function, which is discretized by 10^3 meshes in $x \in [0,1]$. (A). The case of $\alpha = 3$. The center panel shows that the time-average payoff in the first-price auction $\bar{w}^{1\text{st}}(T)$ (green) converges to a lower value than in the second-price auction $\bar{w}^{2\text{nd}}(T)$ (gray), meaning that dissipation of the payoff occurs in the first-price auction. The right panel shows that the time-average revenue in the first-price auction $\bar{R}^{1\text{st}}(T)$ (blue) converges to a higher value than in the second-price auction $\bar{R}^{2\text{nd}}(T)$ (gray), meaning that a gain in revenue occurs in the first-price auction. (B) The case of $\alpha = 0.2$. The dissipation occurs in the time-average payoff in the first-price auction (center) again, but the loss also occurs in the time-average revenue (right).

$\Delta R_{m^*(t)}(m(t))$ is divided as

$$\Delta R_{m^*(t)}(m(t)) \tag{29}$$
$$= (m(t) - m^*(t))\Delta w_{m^*(t)}(m(t)) \tag{30}$$
$$- (m(t)+1)\Delta w_{m^*(t)}(m(t)) \tag{31}$$
$$= -m(t)(m(t)+1)\dot{m}(t) + \mathrm{RevenueGap}(t). \tag{32}$$

Here, since $\Delta w_{m^*(t)}(m(t))$ is proportional to $m(t)-m^*(t)$, the first term is proportional to $(m(t)-m^*(t))^2$. By applying the proof of Thm. 4.1, the second term is also proportional to $(m(t)-m^*(t))^2$ in its time average. By further direct calculation, we can prove the theorem. □

Corollary 4.4 (Revenue Gain). *If $F(x)$ is an inverse polynomial function with $k < 1$ for all $k \in K$ and has only positive coefficients, $\Delta\bar{R}(\infty) > 0$ holds under Asm. 1.*

Proof. Since $(0 <)k < 1$ for all $k \in K$ with $a_k > 0$, the term of RevenueGap(t) is always positive, leading to $\Delta\bar{R}(\infty) > 0$. □

Corollary 4.5 (Revenue Loss). *If $F(x)$ is an inverse polynomial function with $k > 1$ for all $k \in K$ and has only positive coefficients, $\Delta\bar{R}(\infty) < 0$ holds under Asm. 1.*

Proof. Since $k > 1$ for all $k \in K$ with $a_k > 0$, the term of RevenueGap(t) is always negative, leading to $\Delta\bar{R}(\infty) < 0$. □

Remark on the corollaries: These corollaries are applicable in examples often used in auction theory, even though we restricted the range of k to obtain the corollaries. Indeed, Cor. 4.4 is applicable when F is the power distribution with $\alpha > 1$. Also, Cor. 4.5 is applicable when F is the power (with $\alpha < 1$), exponential, or Pareto distribution.

5 EXPERIMENTS

This section experimentally investigate how revenue/payoff equivalence is broken (i.e., Cors. 4.2, 4.4, and 4.5) by the number of bidders varying over time. We examine the power distributions (Exm. 1) to visualize the result of our theorems. We also employ the beta distributions to broaden our experimental scope. The beta distribution is particularly valuable here for two reasons: it allows us to analyze diverse distribution shapes (e.g., unimodal and bimodal) and to verify that our theoretical insights remain qualitatively valid even outside of the strict theoretical assumptions (i.e., Defs. 1 and 2).

5.1 Power Distribution

We now consider the power distributions of $F(x) = x^3$ (Fig. 2-A) and $F(x) = x^{0.2}$ (B), whose $f(x)$ monotonically increase and descrease with x, respectively.

Dissipation in long-run payoff: See the center panels of Fig. 2-A and B. In both of the panels, $\bar{w}^{1\text{st}}(T) < \bar{w}^{2\text{nd}}(T) \Leftrightarrow \Delta\bar{w}(T) < 0$

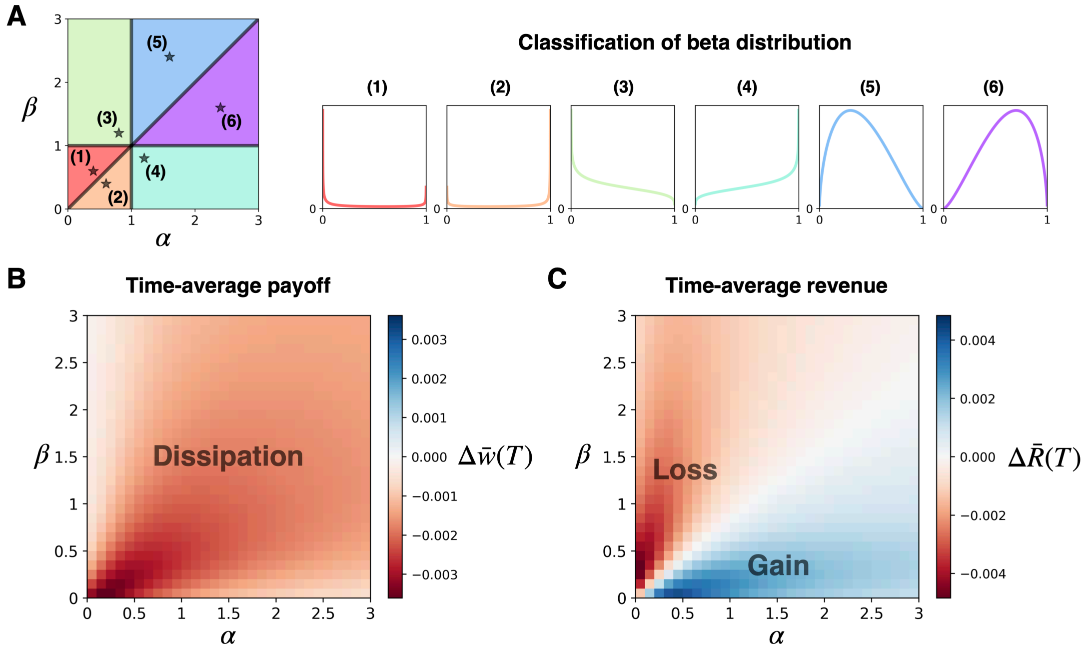

Figure 3: Simulations for the beta distribution. We used the Runge-Kutta fourth-order method with the step size of $1/10$, and learning is accelerated 200 times. We compute the time average until the final time $T = 10^3$. When we calculate the integrals of $S_{m(t),n(t)}$ and $T_{m(t)}$, we approximate $F(x)$ by a step function, which is discretized by 10^3 meshes for $x \in [0,1]$. (A). Categorization of the beta distributions. The left panel shows the six categories. The six right panels show their typical examples. (B). $\Delta\bar{w}(T)$, i.e., the difference of the time-average payoffs between the first-price and second-price auctions. The white color indicates $\Delta\bar{w}(T) = 0$ (payoff equivalence). The darker blue (resp. red) shows that the $\Delta\bar{w}(T)$ is more largely positive (resp. negative). Thus, $\Delta\bar{w}(T) < 0$ holds for all α and β. (C). $\Delta\bar{R}(T)$, i.e., the difference of the time-average revenues between the first-price and second-price auctions. The white color indicates $\Delta\bar{R}(T) = 0$ (revenue equivalence). The darker blue (resp. red) shows that the $\Delta\bar{R}(T)$ is more largely positive (resp. negative).

holds for large T. Thus, we observe that dissipation occurs in the long-run payoff in the first-price auction, consistent with Cor. 4.2.

Gain or loss in long-run revenue: Compare the right panels of Fig. 2-A and B. In A, the power distribution with $\alpha = 3$ is expressed as $a_k = 1$ for $k = 1/3$ in the notation of Def. 1. Since $k < 1$ is satisfied, Cor. 4.4 holds. Indeed, we see that $\bar{R}^{1\text{st}}(T) > \bar{R}^{2\text{nd}}(T) \Leftrightarrow \Delta\bar{w}(T) > 0$ holds for large T, meaning a gain in long-run revenue is generated by the time-varying number of bidders. On the other hand, in B, $\alpha = 0.2$ is expressed as $a_k = 1$ for $k = 5$. Since $k > 1$ is satisfied, Cor. 4.5 holds. Indeed, we see that $\bar{R}^{1\text{st}}(T) < \bar{R}^{2\text{nd}}(T) \Leftrightarrow \Delta\bar{w}(T) < 0$ holds for large T, meaning a loss in long-run revenue.

6 BETA DISTRIBUTION

We further provide the experiments on the beta distribution (of the first kind), which is defined as

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}. \tag{33}$$

First, Fig. 3-A shows that the beta distribution generates various shapes of functions depending on (α,β). The cases of (1) and (2) correspond to bimodal functions. The cases of (3) and (4) correspond to monotonic increasing and decreasing functions, respectively. The cases of (5) and (6) correspond to unimodal functions, such as normal and log-normal distributions. Furthermore, depending on which of α or β is larger, whether the distribution is biased in the side of $x < 1/2$ or $x > 1/2$ changes (compare (1) with (2), (3) with (4), and (5) with (6)).

Dissipation in long-run payoff: Fig. 3-B shows that $\Delta\bar{w}(T) < 0$ holds at final T for all α and β. Thus, this demonstrates that the time-varying number of bidders generates a dissipation in the long-run payoff in various shapes of probability distributions.

Gain or loss in long-run revenue: Fig. 3-C shows that $\Delta\bar{R}(T) > 0$ holds at final T mainly when $\beta < \alpha$ (blue area), whereas $\Delta\bar{R}(T) < 0$ holds mainly when $\alpha < \beta$ (red area). Thus, the time-varying number of bidders generates a gain and loss in the long-run revenue depending on the cases of $\beta < \alpha$ and $\alpha < \beta$, respectively. This is interpreted as follows. First, $\beta < \alpha$ means that the probability distribution f is biased on the side of $x = x_{\mathrm{M}}(= 1)$ like the cases of (2), (4), and (6). Thus, $F(x)$ slowly increases with x and is similar to the power distribution with $\alpha > 1$. Thus, the result of Cor. 4.4 intuitively holds as similar to the case of Fig. 2-A. On the other hand, $\alpha < \beta$ means that the probability distribution is biased on the side of $x = 0$. Thus, $F(x)$ rapidly increases with x and is similar to the power distribution with $\alpha < 1$. Thus, the result of Cor. 4.5 intuitively holds as similar to the case of Fig. 2-C.

7 CONCLUSION

This study considers the setting of symmetric bidders with private and independent values and extends it to multi-agent learning. Under Asm. 1 and Def. 1, we compared the time-average payoff (Thm. 4.1) and revenue (Thm. 4.3) between first-price and second-price auctions. Under Def. 2, which includes various distributions such as the power (Exm. 1), exponential (Exm. 2), and Pareto (Exm. 3) ones, we found that first-price auctions always generate a loss in the payoff of each bidder (Cor. 4.2). In addition, they generate a gain (Cor. 4.4) or loss (Cor. 4.5) depending on the case. These findings show that payoff and revenue equivalences are broken by the time-varying number of bidders. The numerical calculations for the power distribution are consistent with our theorems, and those for the beta distribution broaden the applicability of the results of this study. This study provided theoretical insight into complex problem, i.e., revenue equivalence in a time-varying environment, which advances auction theory.

REFERENCES

[1] Miguel Alcobendas and Robert Zeithammer. 2022. Adjustment of Bidding Strategies After a Switch to First-Price Rules. In *EC*. 296–296.

[2] James P Bailey and Georgios Piliouras. 2019. Multi-Agent Learning in Network Zero-Sum Games is a Hamiltonian System. In *AAMAS*. 233–241.

[3] Daan Bloembergen, Karl Tuyls, Daniel Hennes, and Michael Kaisers. 2015. Evolutionary dynamics of multi-agent learning: A survey. *Journal of Artificial Intelligence Research* 53 (2015), 659–697.

[4] Shuchi Chawla, Jason D Hartline, David L Malec, and Balasubramanian Sivan. 2010. Multi-parameter mechanism design and sequential posted pricing. In *STOC*.

[5] Xiaotie Deng, Xinyan Hu, Tao Lin, and Weiqiang Zheng. 2022. Nash convergence of mean-based learning algorithms in first price auctions. In *WWW*. 141–150.

[6] Stylianos Despotakis, Ramamoorthi Ravi, and Amin Sayedi. 2021. First-price auctions in online display advertising. *Journal of Marketing Research* 58, 5 (2021), 888–907.

[7] Ulf Dieckmann and Richard Law. 1996. The dynamical theory of coevolution: a derivation from stochastic ecological processes. *Journal of Mathematical Biology* 34 (1996), 579–612.

[8] Yi Feng, Hu Fu, Qun Hu, Ping Li, Ioannis Panageas, Xiao Wang, et al. 2023. On the last-iterate convergence in time-varying zero-sum games: Extra gradient succeeds where optimism fails. In *NeurIPS*. 21933–21944.

[9] Tanner Fiez, Ryann Sim, Stratis Skoulakis, Georgios Piliouras, and Lillian Ratliff. 2021. Online learning in periodic zero-sum games. In *NeurIPS*, Vol. 34. 10313–10325.

[10] Yuma Fujimoto, Kaito Ariu, and Kenshi Abe. 2025. Global Behavior of Learning Dynamics in Zero-Sum Games with Memory Asymmetry. In *AAMAS*.

[11] Yuma Fujimoto, Kaito Ariu, and Kenshi Abe. 2025. Synchronization in Learning in Periodic Zero-Sum Games Triggers Divergence from Nash equilibrium. In *AAAI*.

[12] Yuma Fujimoto, Kaito Ariu, and Kenshi Abe. 2026. Time-Varyingness in Auction Breaks Revenue Equivalence. In *AAMAS*.

[13] Shumpei Goke, Gabriel Y Weintraub, Ralph A Mastromonaco, and Samuel S Seljan. 2022. Bidders' Responses to Auction Format Change in Internet Display Advertising Auctions. In *EC*. 295–295.

[14] John C Harsanyi. 1967. Games with incomplete information played by "Bayesian" players, I–III Part I. The basic model. *Management science* 14, 3 (1967), 159–182.

[15] John C Harsanyi. 1968. Games with incomplete information played by "Bayesian" players part II. Bayesian equilibrium points. *Management science* 14, 5 (1968), 320–334.

[16] Ronald M Harstad, John H Kagel, and Dan Levin. 1990. Equilibrium bid functions for auctions with an uncertain number of bidders. *Economics Letters* 33, 1 (1990), 35–40.

[17] Josef Hofbauer and Karl Sigmund. 1990. Adaptive dynamics and evolutionary stability. *Applied Mathematics Letters* 3, 4 (1990), 75–79.

[18] René Kirkegaard. 2012. A mechanism design approach to ranking asymmetric auctions. *Econometrica* 80, 5 (2012), 2349–2364.

[19] Yoav Kolumbus and Noam Nisan. 2022. Auctions between regret-minimizing agents. In *WWW*. 100–111.

[20] Vijay Krishna. 2009. *Auction theory.* Academic press.

[21] Eric Maskin and John Riley. 2000. Asymmetric auctions. *The Review of Economic Studies* 67, 3 (2000), 413–438.

[22] Panayotis Mertikopoulos, Christos Papadimitriou, and Georgios Piliouras. 2018. Cycles in adversarial regularized learning. In *SODA*. 2703–2717.

[23] Panayotis Mertikopoulos and William H Sandholm. 2016. Learning in games via reinforcement and regularization. *Mathematics of Operations Research* 41, 4 (2016), 1297–1324.

[24] Roger B Myerson. 1981. Optimal auction design. *Mathematics of operations research* 6, 1 (1981), 58–73.

[25] Toshihiro Ota and Yuma Fujimoto. 2025. The Hamiltonian of Poly-matrix Zero-sum Games. *arXiv preprint arXiv:2505.12609* (2025).

[26] Renato Paes Leme, Georgios Piliouras, Jon Schneider, Kelly Spendlove, and Song Zuo. 2024. Complex dynamics in autobidding systems. In *EC*.

[27] Renato Paes Leme, Balasubramanian Sivan, and Yifeng Teng. 2020. Why do competitive markets converge to first-price auctions?. In *WWW*. 596–605.

[28] John G Riley and William F Samuelson. 1981. Optimal auctions. *The American Economic Review* 71, 3 (1981), 381–392.

[29] Yuzuru Sato, Eizo Akiyama, and J Doyne Farmer. 2002. Chaos in learning a simple two-person game. *Proceedings of the National Academy of Sciences* 99, 7 (2002), 4748–4751.

[30] William Vickrey. 1961. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance* 16, 1 (1961), 8–37.

[31] Shuai Yuan, Jun Wang, Bowei Chen, Peter Mason, and Sam Seljan. 2014. An empirical study of reserve price optimisation in real-time bidding. In *SIGKDD*. 1897–1906.

[32] Shuai Yuan, Jun Wang, and Xiaoxue Zhao. 2013. Real-time bidding for online advertising: measurement and analysis. In *ADKDD*. 1–8.

Appendix

A NOTATION

This section lists all the notations used in our results (after Sec. 3.1).

- $t \in [0, T]$: time
- $n(t) \in \mathbb{N}$: the number of bidders
- $m(t) \in \mathbb{R}$: the estimated number of the other bidders
- $m^*(t) = n(t) - 1$: the true number of the other bidders
- $x \in [0, x_{\mathrm{M}}]$: value of an item
- $f(x)$, $F(x)$: probability distribution, cumulative distribution of value
- $b(x; m(t))$: bidding in first-price auction
- $G(x) = F(x)^{n-1}$: probability that all the others' values are lower than x
- $w^{\mathrm{1st}}_{m^*(t)}(m(t))$, $w^{\mathrm{2nd}}_{m^*(t)}(m(t))$: payoff in first-price, second-price auctions
- $R^{\mathrm{1st}}_{m^*(t)}(m(t))$, $R^{\mathrm{2nd}}_{m^*(t)}(m(t))$: revenue in first-price, second-price auctions
- $\Delta w_{m^*(t)}(m(t))$, $\Delta R_{m^*(t)}(m(t))$: the difference of payoff, revenue between first-price and second-price auctions
- $\bar{w}^{\mathrm{1st}}(T)$, $\bar{w}^{\mathrm{2nd}}(T)$: time-average payoff in first-price, second-price auctions
- $\bar{R}^{\mathrm{1st}}(T)$, $\bar{R}^{\mathrm{2nd}}(T)$: time-average revenue in first-price, second-price auctions
- $\Delta\bar{w}(T)$, $\Delta\bar{R}(T)$: time-average of payoff, revenue difference
- $P(F)$: the inverse function of F (i.e., $P(F(x)) = x$)
- k, a_k: the power, coefficient of the polynomial function
- α, β, λ: the hyperparameters of a specific class of F

B DERIVATIONS

B.1 Detailed Calculation of Learning Dynamics

First, we consider $w(m + \delta m, m)$, showing that the focal bidder increases its bidding by increasing its estimation of the number of the others. Here, we consider that the win rate changes from $G(x)$ to $G(x + \delta x)$. By assuming $|\delta m| \ll 1$ is sufficiently small and ignoring higher order terms than δm, we expand $w(m + \delta m, m)$ as

$$w_{m^*}(m', m) = \int_0^{x_{\mathrm{M}}} (x - b(x; m')) f(x) F(x')^{m^*} \mathrm{d}x \tag{A1}$$

Here, we defined $x' = x'(x, m, m')$ such that $b(x'; m) = b(x; m')$, meaning that when the focal bidder whose estimation is m' observes the item value x, it bids as if it observes v' for the others whose estimations are m.

$$\begin{aligned} w(m + \delta m, m) &= \int_0^{x_{\mathrm{M}}} (x - b(x; m + \delta m)) f(x) G(x + \delta x) \mathrm{d}x & \text{(A2)} \\ &= w(m, m) + \int_0^{x_{\mathrm{M}}} \underbrace{-\frac{\partial b(x; m)}{\partial m} f(x) G(x)}_{\text{Loss by increasing payment}} \delta m + \underbrace{(x - b(x; m)) f(x) g(x)}_{\text{Gain by increasing win rate}} \delta x \mathrm{d}x. & \text{(A3)} \end{aligned}$$

Here, the former term in the integral means that by the focal bidder escalating its bidding, a payoff loss occurs due to the increase in the payment. The latter term means that a payoff gain also occurs due to the increase in the win rate. Now, the relation between δm and δx is determined by

$$\frac{\partial b(x; m)}{\partial x} \left.\frac{\partial x'}{\partial m'}\right|_{m=m'} = \frac{\partial b(x; m)}{\partial m}. \tag{A4}$$

Here, by the definition of $b(x; m)$, the gradients of $\partial b(x; m)/\partial m$ and $\partial b(x; m)/\partial x$ are directly calculated as

$$\begin{aligned} \frac{\partial b(x; m)}{\partial m} &= \frac{1}{F(x)^m} \int_0^x F(z)^m (\log F(x) - \log F(z)) \mathrm{d}z, & \text{(A5)} \\ \frac{\partial b(x; m)}{\partial x} &= m \frac{f(x)}{F(x)} \frac{1}{F(x)^m} \int_0^x F(z)^m \mathrm{d}z = m \frac{f(x)}{F(x)} (x - b(x; m)). & \text{(A6)} \end{aligned}$$

Finally, we obtain

$$\left.\frac{\partial w_{m^*}(m',m)}{\partial m'}\right|_{m'=m} = \int_0^{x_{\mathrm{M}}} -\frac{\partial b(x;m)}{\partial m} f(x)F(x)^{m^*} + m^*(x-b(x;m))f(x)^2F(x)^{m^*-1} \left.\frac{\partial x'}{\partial m'}\right|_{m'=m} \mathrm{d}x \tag{A7}$$

$$= \frac{m^*-m}{m}\int_0^{x_{\mathrm{M}}} f(x)F(x)^{m^*-m}\int_0^x F(z)^m(\log F(x)-\log F(z))\mathrm{d}z\mathrm{d}x \tag{A8}$$

$$= \frac{m^*-m}{m(m^*-m+1)}\left[F(x)^{m^*-m+1}\int_0^x F(z)^m(\log F(x)-\log F(z))\mathrm{d}z\right]_0^{x_{\mathrm{M}}} \tag{A9}$$

$$-\frac{m^*-m}{m(m^*-m+1)}\int_0^{x_{\mathrm{M}}} f(x)F(x)^{n-m-1}\int_0^x F(z)^m\mathrm{d}z\mathrm{d}x \tag{A10}$$

$$= \frac{m^*-m}{m(m^*-m+1)}\left[F(x)^{m^*-m+1}\int_0^x F(z)^m(\log F(x)-\log F(z))\mathrm{d}z\right]_0^{x_{\mathrm{M}}} \tag{A11}$$

$$-\frac{m^*-m}{m(m^*-m+1)^2}\left[F(x)^{m^*-m+1}\int_0^x F(z)^m\mathrm{d}z\right]_0^{x_{\mathrm{M}}} + \frac{m^*-m}{m(m^*-m+1)^2}\int_0^{x_{\mathrm{M}}} F(x)^{m^*+1}\mathrm{d}x \tag{A12}$$

$$= \frac{m^*-m}{m(m^*-m+1)}\left(\underbrace{-\int_0^{x_{\mathrm{M}}} F(x)^m\log F(x)\mathrm{d}x}_{=T_m} - \underbrace{\frac{1}{m^*-m+1}\int_0^{x_{\mathrm{M}}} F(x)^m - F(x)^{m^*+1}\mathrm{d}x}_{=S_{m,m^*+1}}\right) \tag{A13}$$

$$= -\frac{m^*-m}{m(m^*-m+1)}(T_m - S_{m,m^*+1}). \tag{A14}$$

$$\left.\frac{\partial w(m',m)}{\partial m'}\right|_{m'=m} = \frac{w(m+\delta m,m)-w(m,m)}{\delta m} \tag{A15}$$

$$= \int_0^{x_{\mathrm{M}}} -\frac{\partial b(x;m)}{\partial m} f(x)G(x) + (x-b(x;m))f(x)g(x)\frac{\delta x}{\delta m}\mathrm{d}x \tag{A16}$$

$$= \frac{n-m-1}{m}\int_0^{x_{\mathrm{M}}} f(x)F(x)^{n-m-1}\int_0^x F(z)^m(\log F(x)-\log F(z))\mathrm{d}z\mathrm{d}x \tag{A17}$$

$$= \frac{n-m-1}{m(n-m)}\left[F(x)^{n-m}\int_0^x F(z)^m(\log F(x)-\log F(z))\mathrm{d}z\right]_0^{x_{\mathrm{M}}} \tag{A18}$$

$$-\frac{n-m-1}{m(n-m)}\int_0^{x_{\mathrm{M}}} f(x)F(x)^{n-m-1}\int_0^x F(z)^m\mathrm{d}z\mathrm{d}x \tag{A19}$$

$$= \frac{n-m-1}{m(n-m)}\left[F(x)^{n-m}\int_0^x F(z)^m(\log F(x)-\log F(z))\mathrm{d}z\right]_0^{x_{\mathrm{M}}} \tag{A20}$$

$$-\frac{n-m-1}{m(n-m)^2}\left[F(x)^{n-m}\int_0^x F(z)^m\mathrm{d}z\right]_0^{x_{\mathrm{M}}} + \frac{n-m-1}{m(n-m)^2}\int_0^{x_{\mathrm{M}}} F(x)^n\mathrm{d}x \tag{A21}$$

$$= \underbrace{\frac{n-m-1}{m(n-m)}}_{=-\frac{m-m^*}{m(n-m)}}\left(\underbrace{-\int_0^{x_{\mathrm{M}}} F(x)^m\log F(x)\mathrm{d}x}_{=T_m} - \underbrace{\frac{1}{n-m}\int_0^{x_{\mathrm{M}}} F(x)^m - F(x)^n\mathrm{d}x}_{=S_{m,n}}\right) \tag{A22}$$

$$= -\frac{m-m^*}{m(n-m)}(T_m - S_{m,n}). \tag{A23}$$

This corresponds to Eq. (7).

B.2 Detailed Calculation for Convergence

We rewritten Eq. (7) as

$$\dot{m} = -\frac{m-m^*}{m}\int_0^{x_{\mathrm{M}}} F(x)^m H(F(x))\mathrm{d}x, \tag{A24}$$

$$H(F) := -\frac{1}{n-m}\log F - \frac{1}{(n-m)^2}(1-F^{n-m}). \tag{A25}$$

Here, $H(F) > 0$ always holds for $0 < F < 1$ because $H(1) = 0$ and

$$\frac{\mathrm{d}H}{\mathrm{d}F} = -\frac{1}{(n-m)F}(1 - F^{n-m}) < 0, \tag{A26}$$

for all $0 < F < 1$ and $(n-m) \in \mathbb{R}$. Here, we remark the limit of $n - m \to 0$ exists. Thus, we prove $\dot{m} < 0$ for $m > m^*$ and $\dot{m} > 0$ for $m < m^*$, meaning the convergence to the equilibrium $m = m^*$.

B.3 Detailed Calculation of Payoff

When all the bidders use $b(x; m)$, their payoffs are calculated as

$$\begin{aligned}
w^{\mathrm{1st}}(m) &= w(m, m) && \text{(A27)}\\
&= \int_0^{x_{\mathrm{M}}} (x - b(x; m)) f(x) G(x) \mathrm{d}x && \text{(A28)}\\
&= \int_0^{x_{\mathrm{M}}} f(x) F(x)^{n-m-1} \int_0^x F(z)^m \mathrm{d}z \mathrm{d}x && \text{(A29)}\\
&= \frac{1}{n-m}\left(\left[F(x)^{n-m}\int_0^x F(z)^m \mathrm{d}z\right]_0^{x_{\mathrm{M}}} - \int_0^{x_{\mathrm{M}}} F(x)^n \mathrm{d}x\right) && \text{(A30)}\\
&= \frac{1}{n-m}\int_0^{x_{\mathrm{M}}} F(x)^m - F(x)^n \mathrm{d}x && \text{(A31)}\\
&= S_{m,n}. && \text{(A32)}
\end{aligned}$$

This corresponds to Eq. (11).

C PROOFS

C.1 Proof of Thm. 4.1

Proof. Assume $\sum_{k\in K} a_k F(x)^k = x$, and then we obtain

$$f(x) \sum_{k\in K} a_k k F(x)^{k-1} = 1. \tag{A33}$$

By using this equation, we calculate $S_{m,n}$ and T_m as

$$\begin{aligned}
S_{m,n} &= \frac{1}{n-m}\int_0^{x_{\mathrm{M}}} F(x)^m - F(x)^n \mathrm{d}x && \text{(A34)}\\
&= \frac{1}{n-m}\int_0^{x_{\mathrm{M}}} f(x)\sum_{k\in K} a_k k F(x)^{k-1}(F(x)^m - F(x)^n)\mathrm{d}x && \text{(A35)}\\
&= \sum_{k\in K} a_k k \frac{1}{n-m}\int_0^{x_{\mathrm{M}}} f(x)(F(x)^{m+k-1} - F(x)^{n+k-1})\mathrm{d}x && \text{(A36)}\\
&= \sum_{k\in K} a_k k \frac{1}{n-m}\left[\frac{1}{m+k}F(x)^{m+k} - \frac{1}{n+k}F(x)^{n+k}\right]_0^{x_{\mathrm{M}}} && \text{(A37)}\\
&= \sum_{k\in K} a_k k \frac{1}{(m+k)(n+k)}, && \text{(A38)}\\
T_m &= -\int_0^{x_{\mathrm{M}}} F(x)^m \log F(x) \mathrm{d}x && \text{(A39)}\\
&= -\int_0^{x_{\mathrm{M}}} f(x)\sum_{k\in K} a_k k F(x)^{k-1} F(x)^m \log F(x)\mathrm{d}x && \text{(A40)}\\
&= -\sum_{k\in K} a_k k \int_0^{x_{\mathrm{M}}} f(x) F(x)^{m+k-1}\log F(x)\mathrm{d}x && \text{(A41)}\\
&= -\sum_{k\in K} a_k k \left[\frac{1}{m+k}F(x)^{m+k}\log F(x) - \frac{1}{(m+k)^2}F(x)^{m+k}\right]_0^{x_{\mathrm{M}}} && \text{(A42)}\\
&= \sum_{k\in K} a_k k \frac{1}{(m+k)^2}. && \text{(A43)}
\end{aligned}$$

Using T_m and $S_{m,n}$, we calculate and $m\dot{m}$ as

$$
\begin{aligned}
m\dot{m} &= -\frac{m-m^*}{n-m}(T_m - S_{m,n}) && \text{(A44)}\\
&= -\frac{m-m^*}{n-m}\sum_{k\in K} a_k k\left(\frac{1}{(m+k)^2} - \frac{1}{(m+k)(n+k)}\right) && \text{(A45)}\\
&= -\frac{m-m^*}{n-m}\sum_{k\in K} a_k k\frac{n-m}{(m+k)^2(n+k)} && \text{(A46)}\\
&= -(m-m^*)\sum_{k\in K}\frac{a_k k}{(m+k)^2(n+k)}. && \text{(A47)}
\end{aligned}
$$

Also, we calculate $\Delta w(m)$ as

$$
\begin{aligned}
\Delta w(m) &= S_{m,n} - S_{m^*,n} && \text{(A48)}\\
&= \sum_{k\in K} a_k k\left(\frac{1}{(m+k)(n+k)} - \frac{1}{(m^*+k)(n+k)}\right) && \text{(A49)}\\
&= \sum_{k\in K} a_k k\frac{-m+m^*}{(m+k)(m^*+k)(n+k)} && \text{(A50)}\\
&= -(m-m^*)\sum_{k\in K}\frac{a_k k}{(m+k)(m^*+k)(n+k)}. && \text{(A51)}
\end{aligned}
$$

By these equations, we obtain the following relationship:

$$
\begin{aligned}
\Delta w(m) &= m\dot{m} + (m-m^*)\sum_{k\in K}\frac{a_k k}{(m+k)(n+k)}\left(\frac{1}{m+k} - \frac{1}{m^*+k}\right) && \text{(A52)}\\
&= m\dot{m} - \underbrace{(m-m^*)^2\sum_{k\in K}\frac{a_k k}{(m+k)^2(m^*+k)(n+k)}}_{\text{Dissipation}}. && \text{(A53)}
\end{aligned}
$$

Its time average is computed as

$$
\begin{aligned}
\Delta\bar{w} &= \underbrace{\lim_{T\to\infty}\frac{1}{T}\int_0^T m\dot{m}\mathrm{d}t}_{=0} - \lim_{T\to\infty}\frac{1}{T}\int_0^T \text{Dissipation d}t && \text{(A54)}\\
&= -\lim_{T\to\infty}\frac{1}{T}\int_0^T \text{Dissipation d}t, && \text{(A55)}
\end{aligned}
$$

where the first term is negligible by the boundedness of m. We have proved the theorem. □

C.2 Proof of Thm. 4.3

Proof. We calculate $\Delta R(m)$ as

$$
\begin{aligned}
\Delta R(m) =& -n\Delta w(m) && \text{(A56)}\\
=&(m-m^*)\Delta w(m) - (m+1)\Delta w(m) && \text{(A57)}\\
=&(m-m^*)^2\sum_{k\in K}\frac{a_k k}{(m+k)(m^*+k)(n+k)} && \text{(A58)}\\
&-(m+1)\left\{m\dot{m} - (m-m^*)^2\sum_{k\in K}\frac{a_k k}{(m+k)^2(m^*+k)(n+k)}\right\} && \text{(A59)}\\
=& -m(m+1)\dot{m} + (m-m^*)^2\sum_{k\in K}\frac{-a_k k}{(m+k)(m^*+k)(n+k)}\left(1-\frac{m+1}{m+k}\right) && \text{(A60)}\\
=& -m(m+1)\dot{m} + \underbrace{(m-m^*)^2\sum_{k\in K}\frac{-a_k k(k-1)}{(m+k)^2(m^*+k)(n+k)}}_{\text{RevenueGap}}. && \text{(A61)}
\end{aligned}
$$

In the third equal sign, we used Eq. (A53). Its time average is computed as

$$\Delta\bar{R} = \underbrace{-\lim_{T\to\infty}\frac{1}{T}\int_0^T m(m+1)\dot{m}\mathrm{d}t}_{=0} + \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{RevenueGap}\ \mathrm{d}t \tag{A62}$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{RevenueGap}\ \mathrm{d}t, \tag{A63}$$

where the first term is negligible by the boundedness of m. We have proved the theorem. □